

SYMPLECTIC ORIGAMI

A. CANNAS DA SILVA, V. GUILLEMIN, AND A. R. PIRES

ABSTRACT. An origami manifold is a manifold equipped with a closed 2-form which is symplectic except on a hypersurface where it is like the pullback of a symplectic form by a folding map and its kernel defines a circle fibration. We can move back and forth between origami and symplectic manifolds using cutting (unfolding) and radial blow-up (folding), modulo compatibility conditions. We prove an origami convexity theorem for hamiltonian torus actions, classify toric origami manifolds by polyhedral objects resembling paper origami and discuss examples. We also prove a cobordism result and some of its classical consequences, and compute the cohomology of a special class of origami manifolds.

1. INTRODUCTION

This is the third in a series of papers on *folded* symplectic manifolds. The first of these papers [CGW] contains a description of the basic local and semi-global features of these manifolds and of the *folding* and *unfolding* operations; in the second [C] it is shown that a manifold is folded symplectic if and only if it is stable complex (and in particular that every oriented 4-manifold is folded symplectic).¹

In this third paper we take up the theme of hamiltonian group actions on folded symplectic manifolds. In particular, we focus on a special class of folded symplectic manifolds which we call *origami manifolds*.²

Date: September 22, 2009.

The first author was partially supported by the Fundação para a Ciência e a Tecnologia (FCT/Portugal). The third author was supported by FCT grant SFRH/BD/21657/2005.

¹Other recent papers on the topology of folded symplectic manifolds are [Ba] and [vB].

²Jean-Claude Hausmann pointed out to us that the term “origami” had once been proposed for another class of spaces: what are now known as orbifolds.

For the purposes of this introduction, let us say that a folded symplectic manifold is a triple (M, Z, ω) where M is an oriented $2n$ -dimensional manifold, ω a closed 2-form and $Z \xrightarrow{i} M$ a hypersurface. “Folded symplectic” requires that ω be symplectic on $M \setminus Z$ and that the restriction of ω to Z be odd-symplectic, i.e.

$$(i^*\omega)^{n-1} \neq 0 .$$

From this one gets on Z a *null foliation* by lines and (M, Z, ω) is “origami” if this foliation is fibrating with compact connected fibers. In this case one can *unfold* M by taking the closures of the connected components of $M \setminus Z$ and identifying boundary points on the same leaf of the null foliation. We will prove that this unfolding defines a cobordism between (a compact) M and a disjoint union of (compact) symplectic manifolds M_i :

$$(1) \quad M \sim \bigsqcup_i M_i .$$

Moreover, if M is a hamiltonian G -manifold we will prove that the M_i ’s are as well. The *origami* results of this paper involve reconstructing the moment data of M (and in the toric case M itself) from the moment data of the M_i ’s.

Precise definitions of “folded symplectic” and “origami” are given in Section 2.1. In 2.2 we describe in detail the unfolding operation (1) and in 2.3 how one can refold the terms on the right to reconstruct M via a radial blow-up operation. Then in Sections 2.4 and 2.5 we prove that folding and unfolding are inverse operations: unfolding followed by folding gives one the manifold one started with and vice-versa.

With these preliminaries out of the way, we turn in Section 3 to the main theme of this paper: torus actions on origami manifolds. In 3.1 we define for such actions an origami version of the notion of moment polytope, which turns out to be a collection of convex polytopes with compatibility conditions, or *folding instructions* on facets. We then concentrate in Section 3.2 on the toric case and prove in 3.3 an origami version of the Delzant theorem. More explicitly, we show that toric origami manifolds are classified by *origami templates*: triples $(\mathcal{P}, \mathcal{F}, \mathcal{O})$, where \mathcal{P} is a finite collection of n -dimensional Delzant polytopes and \mathcal{F} a collection of pairs of facets of these polytopes satisfying

- (a) for each pair of facets $\{F_1, F_2\} \in \mathcal{F}$ the corresponding polytopes in \mathcal{P} are identical in a neighborhood of these facets;

- (b) if a facet occurs in a pair, none of its neighboring facets occurs in any other pair;
- (c) the topological space constructed from the disjoint union of all the $\Delta_i \in \mathcal{P}$ by identifying facet pairs in \mathcal{F} is connected and oriented according to \mathcal{O} .

Without the assumption that M be origami, i.e., that the null foliation be fibrating, it is *not* possible to classify hamiltonian torus actions on folded symplectic manifolds by a finite set of combinatorial data; why not is illustrated by example 3.11. Nonetheless, Chris Lee has shown that a (more intricate) classification of these objects by moment data *is* possible at least in dimension four [Lee]. This result of his we found very helpful in putting our own results into perspective.

In Section 4 we prove that (1) is a cobordism and, in fact, an equivariant cobordism in presence of group actions. We then discuss some consequences of this result: we recall a theorem from [CGW] which asserts that if M is folded symplectic then it is stable complex, hence if the presymplectic form on M is integrable, M admits a pre-quantum line bundle and a spin- \mathbb{C} structure. Then one can quantize M (equivariantly if a group is acting) and, since cobordant spaces have the same quantization, from (1) we get an isomorphism

$$\mathcal{Q}(M) \simeq \oplus (-1)^{\sigma_i} \mathcal{Q}(M_i)$$

where $\mathcal{Q}(M)$ and $\mathcal{Q}(M_i)$ are the spin- \mathbb{C} quantizations of M and M_i (§8 in [CGW]) and $(-1)^{\sigma_i} = +1$ if the orientation of M_i coincides with the symplectic orientation of $M \setminus Z$ and -1 otherwise. This also implies (§2 in [GGK]) that if \mathfrak{m} is the Duistermaat-Heckman measure of M and \mathfrak{m}_i that of M_i then

$$(2) \quad \mathfrak{m} = \sum (-1)^{\sigma_i} \mathfrak{m}_i .$$

The final two sections of this paper are devoted to origami versions of two theorems in standard theory of hamiltonian actions. In Section 5 we prove an alternative version of the Duistermaat-Heckman theorem (2) for origami manifolds and show that this gives another way of thinking of a toric origami manifold as a superimposition of polyhedra: in this case, polyhedral cones. In Section 6 we compute the cohomology groups of a toric origami manifold, modulo the assumption that the folding hypersurface be connected.

Throughout this introduction we have been assuming that our origami manifolds are oriented. Except for the cobordism result (1) and

its consequences for quantization, all the results of this paper extend to the case of nonorientable origami manifolds. Moreover, as we show in Section 3, some of the most curious examples of origami manifolds (such as \mathbb{RP}^{2n} and the Klein bottle) are nonorientable.

We would like to express our gratitude to friends and colleagues who have helped us write and rewrite sections of this paper and/or have given us valuable suggestions about the contents. We would particularly like to thank in this regard Yael Karshon, Allen Knutson, Chris Lee, Sue Tolman, Kartik Venkatram and Alan Weinstein.

2. ORIGAMI MANIFOLDS

2.1. Folded symplectic and origami forms.

Definition 2.1. A **folded symplectic form** on a $2n$ -dimensional manifold M is a closed 2-form ω whose top power ω^n vanishes transversally on a submanifold Z and whose restriction to that submanifold has maximal rank. The submanifold Z is necessarily of codimension one, embedded and is called the **folding hypersurface** or **fold**.

An analogue of Darboux's theorem for folded symplectic forms [CGW, M] says that near any point $p \in Z$ there is a coordinate chart centered at p where the form ω is

$$x_1 dx_1 \wedge dy_1 + dx_2 \wedge dy_2 + \dots + dx_n \wedge dy_n .$$

Let (M, ω) be a $2n$ -dimensional *folded symplectic manifold*, that is, a manifold M^{2n} equipped with a folded symplectic form ω . Let $i : Z \hookrightarrow M$ be the inclusion of the folding hypersurface Z . Away from Z , the form ω is nondegenerate, so $\omega^n|_{M \setminus Z} \neq 0$. The induced restriction $i^*\omega$ has a one-dimensional kernel at each point: the line field V on Z , called the **null foliation**. Note that $V = TZ \cap E \subset i^*TM$ where E is the rank 2 bundle over Z whose fiber at each point is the kernel of ω .

When a folded symplectic manifold (M, ω) is an oriented manifold, the complement $M \setminus Z$ decomposes into open subsets M^+ where $\omega^n > 0$ and M^- where $\omega^n < 0$. This induces a coorientation on Z and hence an orientation on Z . From the form $(i^*\omega)^{n-1}$, we obtain an orientation of the quotient bundle $(i^*TM)/E$ and hence an orientation of E . From the orientations of TZ and of E , we obtain an orientation of their intersection, the null foliation V .

We concentrate on the case of fibering null foliation.

Definition 2.2. An **origami manifold** is a folded symplectic manifold (M, ω) whose null foliation integrates to a principal S^1 -fibration, called the **null fibration**, over a compact **base**.³

$$\begin{array}{ccc} S^1 & \hookrightarrow & Z \\ & & \downarrow \pi \\ & & B \end{array}$$

The form ω is called an **origami form**. When the manifold M is oriented, we assume that the principal S^1 -action matches the induced orientation of the null foliation V .

Example 2.3. Consider the unit sphere S^{2n} in euclidean space $\mathbb{R}^{2n+1} \simeq \mathbb{C}^n \times \mathbb{R}$ with coordinates $x_1, y_1, \dots, x_n, y_n, h$. Let ω_0 be the restriction to S^{2n} of $dx_1 \wedge dy_1 + \dots + dx_n \wedge dy_n = r_1 dr_1 \wedge d\theta_1 + \dots + r_n dr_n \wedge d\theta_n$. Then ω_0 is a folded symplectic form. The folding hypersurface is the equator sphere given by the intersection with the plane $h = 0$. The null foliation is the Hopf foliation since

$$\iota_{\frac{\partial}{\partial \theta_1} + \dots + \frac{\partial}{\partial \theta_n}} \omega_0 = -r_1 dr_1 - \dots - r_n dr_n$$

is trivial on Z , hence the null fibration is $S^1 \hookrightarrow S^{2n-1} \rightarrow \mathbb{CP}^{n-1}$. Thus, (S^{2n}, ω_0) is an orientable origami manifold. \diamond

Example 2.4. The standard folded symplectic form ω_0 on $\mathbb{RP}^{2n} = S^{2n}/\mathbb{Z}_2$ is induced by the restriction to S^{2n} of the \mathbb{Z}_2 -invariant form $dx_1 \wedge dx_2 + \dots + dx_{2n-1} \wedge dx_{2n}$ in \mathbb{R}^{2n+1} [CGW]. The folding hypersurface is $\mathbb{RP}^{2n-1} \simeq \{[x_1, \dots, x_{2n}, 0]\}$, the null fibration is the \mathbb{Z}_2 -quotient of the Hopf fibration $S^1 \hookrightarrow \mathbb{RP}^{2n-1} \rightarrow \mathbb{CP}^{n-1}$, and $(\mathbb{RP}^{2n}, \omega_0)$ is a nonorientable origami manifold. \diamond

Definition 2.5. Two (oriented) origami manifolds (M, ω) and (M', ω') are **equivalent** if there is a (orientation-preserving) diffeomorphism $\rho : M \rightarrow \widetilde{M}$ such that $\rho^* \widetilde{\omega} = \omega$.

This notion of equivalence stresses the importance of the null foliation being fibrating, and not the particular choice of principal circle fibration. Let (M, ω) be an (oriented) origami manifold with null fibration $S^1 \hookrightarrow Z \xrightarrow{\pi} B$. The null foliation V is the vertical bundle of π . A choice of a (oriented) trivializing section of V , scaled so that its integral curves all have period 2π , gives a vertical vector field generating a possibly different action of S^1 on Z [CGW, §7].

³It would be natural to extend this definition admitting *Seifert fibrations* and *orbifold bases*.

As in symplectic reduction, the base B of the null fibration is naturally symplectic. Firstly, the form $i^*\omega$ is both horizontal and invariant, hence basic. Let ω_B denote the natural (*reduced*) symplectic form on B satisfying

$$i^*\omega = \pi^*\omega_B .$$

The form ω_B is closed and nondegenerate.

2.2. Cutting. The folding hypersurface Z plays the role of an *exceptional divisor* as it can be *blown-down* to obtain honest symplectic pieces.⁴ This process, called **cutting** (or *blowing-down* or *unfolding*), is essentially symplectic cutting and was described in [CGW, Theorem 7] in the orientable case.

Example 2.6. Cutting the origami manifold (S^{2n}, ω_0) from Example 2.3 produces \mathbb{CP}^n and $\overline{\mathbb{CP}^n}$ each equipped with the same multiple of the Fubini-Study form with total volume equal to that of an original hemisphere, $n!(2\pi)^n$. \diamond

Example 2.7. Cutting the origami manifold $(\mathbb{RP}^{2n}, \omega_0)$ from Example 2.4 produces a single copy of \mathbb{CP}^n . \diamond

Proposition 2.8. [CGW] *Let (M^{2n}, ω) be an oriented origami manifold with null fibration $S^1 \hookrightarrow Z \xrightarrow{\pi} B$.*

Then the unions $M^+ \sqcup B$ and $M^- \sqcup B$ each naturally admits a structure of $2n$ -dimensional symplectic manifold, denoted (M_0^+, ω_0^+) and (M_0^-, ω_0^-) respectively, with ω_0^+ and ω_0^- restricting to ω on M^+ and M^- and with a natural embedding of (B, ω_B) as a symplectic submanifold with projectivised normal bundle isomorphic to $Z \rightarrow B$.

The orientation induced from the original orientation on M matches the symplectic orientation on M_0^+ and is opposite to the symplectic orientation on M_0^- .

Proof. By an adaptation of Moser's trick [CGW, Theorem 1], there is a tubular neighborhood \mathcal{U} of Z with a diffeomorphism $\varphi : Z \times (-\varepsilon, \varepsilon) \rightarrow \mathcal{U}$ such that

$$\varphi^*\omega = p^*i^*\omega + d(t^2 p^*\alpha) ,$$

⁴Origami manifolds may hence be interpreted as *birationally symplectic manifolds*. However, in algebraic geometry the designation *birational symplectic manifolds* was used by Huybrechts [H] in a different context, that of birational equivalence for complex manifolds equipped with a holomorphic nondegenerate two-form.

where $p : Z \times (-\varepsilon, \varepsilon) \rightarrow Z$ is the projection onto the first factor, $i : Z \hookrightarrow M$ is the inclusion, t is the real coordinate on the interval $(-\varepsilon, \varepsilon)$ and α is an S^1 -connection on Z .

Let \mathcal{U}^+ denote $M^+ \cap \mathcal{U} = \varphi(Z \times (0, \varepsilon))$. The diffeomorphism

$$\psi : Z \times (0, \varepsilon^2) \rightarrow \mathcal{U}^+, \quad \psi(x, s) = \varphi(x, \sqrt{s})$$

induces a symplectic form

$$\psi^* \omega = p^* i^* \omega + d(sp^* \alpha) := \nu$$

on $Z \times (0, \varepsilon^2)$ that extends by the same formula to $Z \times (-\varepsilon^2, \varepsilon^2)$.

As in standard symplectic cutting [L], we form the product $(Z \times (-\varepsilon^2, \varepsilon^2), \nu) \times (\mathbb{C}, -\omega_0)$ where $\omega_0 = \frac{i}{2} dz \wedge d\bar{z}$. The product action of S^1 on $Z \times (-\varepsilon^2, \varepsilon^2) \times \mathbb{C}$ by

$$e^{i\theta} \cdot (x, s, z) = (e^{i\theta} \cdot x, s, e^{-i\theta} z)$$

is hamiltonian and $\mu(x, s, z) = s - \frac{|z|^2}{2}$ is the *moment map*.⁵ Zero is a regular value of μ and the corresponding level is a codimension-one submanifold which decomposes as

$$\mu^{-1}(0) = Z \times \{0\} \times \{0\} \bigsqcup \{(x, s, z) \mid s > 0, |z|^2 = 2s\}.$$

Since S^1 acts freely on (each summand above of) $\mu^{-1}(0)$, the quotient $\mu^{-1}(0)/S^1$ is a manifold and the point-orbit map is a principal S^1 -bundle. Moreover, we can view it as

$$\mu^{-1}(0)/S^1 \simeq B \sqcup \mathcal{U}^+.$$

Indeed, B embeds as a codimension-two submanifold via

$$\begin{aligned} j : B &\longrightarrow \mu^{-1}(0)/S^1 \\ \pi(x) &\longmapsto [x, 0, 0] \quad \text{for } x \in Z \end{aligned}$$

and \mathcal{U}^+ embeds as an open dense submanifold via

$$\begin{aligned} j^+ : \mathcal{U}^+ &\longrightarrow \mu^{-1}(0)/S^1 \\ \psi(x, s) &\longmapsto [x, s, \sqrt{2s}]. \end{aligned}$$

⁵We say that the action of a Lie group G on an origami manifold (M, ω) is **hamiltonian** if it admits a **moment map**, $\mu : M \rightarrow \mathfrak{g}^*$, satisfying the same conditions as in the symplectic case:

- μ collects hamiltonian functions, i.e., $d\langle \mu, X \rangle = \iota_{X^\#} \omega$, $\forall X \in \text{Lie}(G) =: \mathfrak{g}$, where $X^\#$ is the vector field generated by X , and
- μ is equivariant with respect to the given action of G on M and the coadjoint action of G on the dual vector space \mathfrak{g}^* .

The symplectic form Ω_{red} on $\mu^{-1}(0)/S^1$ obtained by symplectic reduction is such that the above embeddings of (B, ω_B) and of $(\mathcal{U}^+, \omega|_{\mathcal{U}^+})$ are symplectic.

The normal bundle to $j(B)$ in $\mu^{-1}(0)/S^1$ is the quotient over S^1 -orbits (upstairs and downstairs) of the normal bundle to $Z \times \{0\} \times \{0\}$ in $\mu^{-1}(0)$. This latter bundle is the product bundle $Z \times \{0\} \times \{0\} \times \mathbb{C}$ where the S^1 -action is

$$e^{i\theta} \cdot (x, 0, 0, z) = (e^{i\theta} \cdot x, 0, 0, e^{-i\theta} z) .$$

Projectivizing and taking the S^1 -quotient we naturally get the bundle $Z \rightarrow B$ with the natural isomorphism

$$\begin{array}{ccc} (Z \times \{0\} \times \{0\} \times \mathbb{C}^*)/S^1 \ni [x, 0, 0, re^{i\theta}] & \longmapsto & e^{i\theta}x \in Z \\ \downarrow & & \downarrow \\ (Z \times \{0\} \times \{0\})/S^1 \ni [x, 0, 0] & \longmapsto & \pi(x) \in B . \end{array}$$

By gluing the rest of M^+ along \mathcal{U}^+ , we produce a $2n$ -dimensional symplectic manifold (M_0^+, ω_0^+) with a natural symplectomorphism $\overline{j^+} : M^+ \rightarrow M_0^+ \setminus j(B)$ extending j^+ .

For the other side, the map $\psi_- : Z \times (0, \varepsilon^2) \rightarrow \mathcal{U}^- := M^- \cap \mathcal{U}$, $(x, s) \mapsto \varphi(x, -\sqrt{s})$ reverses orientation and $(\psi_-)^*\omega = \nu$. The base B embeds as a symplectic submanifold of $\mu^{-1}(0)/S^1$ by the previous formula. The embedding

$$\begin{array}{ccc} j^- : & \mathcal{U}^- & \longrightarrow \overline{\mu^{-1}(0)/S^1} \\ & \psi_-(x, s) & \longmapsto [x, s, -\sqrt{2s}] \end{array}$$

is an orientation-reversing symplectomorphism. By gluing the rest of M^- along \mathcal{U}^- , we produce (M_0^-, ω_0^-) with a natural symplectomorphism $\overline{j^-} : M^- \rightarrow M_0^- \setminus j(B)$ extending j^- .

Different initial choices of a Moser model φ for a tubular neighborhood \mathcal{U} of Z yield symplectomorphic manifolds. \square

Remark 2.9. The cutting construction in the previous proof produces a symplectomorphism γ between tubular neighborhoods $\mu^{-1}(0)/S^1$ of the embeddings of B in M_0^+ and M_0^- , comprising $\mathcal{U}^+ \rightarrow \mathcal{U}^-$, $\varphi(x, t) \mapsto \varphi(x, -t)$, and the identity map on B :

$$\begin{array}{ccc} \gamma : & \mu^{-1}(0)/S^1 & \longrightarrow \mu^{-1}(0)/S^1 \\ & [x, s, \sqrt{2s}] & \longmapsto [x, s, -\sqrt{2s}] . \end{array}$$

\diamond

Definition 2.10. The symplectic manifolds (M_0^+, ω_0^+) and (M_0^-, ω_0^-) obtained by cutting are called the **symplectic cut pieces** of the oriented origami manifold (M, ω) and the embedded copies of B are called **centers**.

Cutting may be performed for any *nonorientable* origami manifold (M, ω) by working with its orientable double cover. The double cover involution yields a symplectomorphism from one symplectic cut piece to the other. Hence, we regard these pieces as a trivial double cover (of one of them) and call their \mathbb{Z}_2 -quotient the *symplectic cut space* of (M, ω) . In the case where $M \setminus Z$ is connected, the symplectic cut space is also connected; see Example 2.7.

Definition 2.11. The **symplectic cut space** of an origami manifold (M, ω) is the natural \mathbb{Z}_2 -quotient of the symplectic cut pieces of its orientable double cover.

Notice that, when the original origami manifold is compact, the symplectic cut space is also compact.

2.3. Radial blow-up. We can reverse the cutting procedure using a origami (and simpler) analogue of Gompf's gluing construction [G]. *Radial blow-up* is a local operation on a symplectic tubular neighborhood of a codimension-two symplectic submanifold modeled by the following example.

Example 2.12. Consider the standard symplectic $(\mathbb{R}^{2n}, \omega_0)$ with its standard euclidean metric. Let B be the symplectic submanifold defined by $x_1 = y_1 = 0$ with unit normal bundle N identified with the hypersurface $x_1^2 + y_1^2 = 1$. The map $\beta : N \times \mathbb{R} \rightarrow \mathbb{R}^{2n}$ defined by

$$\beta((p, e^{i\theta}), r) = p + (r \cos \theta, r \sin \theta, 0, \dots, 0) \text{ for } p \in B$$

induces by pullback an origami form on the cylinder $N \times \mathbb{R} \simeq S^1 \times \mathbb{R}^{2n-1}$, namely

$$\beta^* \omega_0 = r dr \wedge d\theta + dx_2 \wedge dy_2 + \dots + dx_n \wedge dy_n .$$

◇

Let (M, ω) be a symplectic manifold with a codimension-two symplectic submanifold B . Let $i : B \hookrightarrow M$ be the inclusion map. Consider the projectivised normal bundle over B

$$\mathcal{N} := \mathbb{P}^+(i^*TM/TB) = \{x \in (i^*TM)/TB, x \neq 0\} / \sim$$

where $\lambda x \sim x$ for $\lambda \in \mathbb{R}^+$. We choose an S^1 -action on the circle bundle \mathcal{N} over B . Let $\varepsilon > 0$.

Definition 2.13. A **blow-up model** for a neighborhood \mathcal{U} of B in (M, ω) is a map

$$\beta : \mathcal{N} \times (-\varepsilon, \varepsilon) \longrightarrow \mathcal{U}$$

which factors as

$$\begin{array}{ccc} \beta : \mathcal{N} \times (-\varepsilon, \varepsilon) & \xrightarrow{\beta_0} & (\mathcal{N} \times \mathbb{C}) / S^1 \xrightarrow{\simeq} \mathcal{U} \\ (x, t) & \longmapsto & [x, t] \end{array}$$

where $e^{i\theta} \cdot (x, t) = (e^{i\theta} \cdot x, te^{-i\theta})$ for $(x, t) \in \mathcal{N} \times \mathbb{C}$ and the second arrow is a bundle diffeomorphism from the image of the prototype map β_0 to \mathcal{U} covering the identity $B \rightarrow B$.

From the properties of β_0 , it follows that:

- (1) the restriction of β to $\mathcal{N} \times (0, \varepsilon)$ is an orientation-preserving diffeomorphism onto $\mathcal{U} \setminus B$;
- (2) $\beta(-x, -t) = \beta(x, t)$;
- (3) the restriction of β to $\mathcal{N} \times \{0\}$ is the bundle projection $\mathcal{N} \rightarrow B$;
- (4) for the vector fields ν generating the vertical bundle of $\mathcal{N} \rightarrow B$ and $\frac{\partial}{\partial t}$ tangent to $(-\varepsilon, \varepsilon)$ we have that $D\beta(\nu)$ intersects zero transversally and $D\beta(\frac{\partial}{\partial t})$ is never zero.

All blow-up models share the same germ up to diffeomorphism. More precisely, if $\beta_1 : \mathcal{N} \times (-\varepsilon, \varepsilon) \rightarrow \mathcal{U}_1$ and $\beta_2 : \mathcal{N} \times (-\varepsilon, \varepsilon) \rightarrow \mathcal{U}_2$ are two blow-up models for neighborhoods \mathcal{U}_1 and \mathcal{U}_2 of B in (M, ω) , then there are possibly narrower tubular neighborhoods of B , $\mathcal{V}_i \subseteq \mathcal{U}_i$ and a diffeomorphism $\gamma : \mathcal{V}_1 \rightarrow \mathcal{V}_2$ such that $\beta_2 = \gamma \circ \beta_1$.

In practice, a blow-up model may be obtained by choosing a riemannian metric to identify \mathcal{N} with the unit bundle inside the geometric normal bundle TB^\perp and then by using the exponential map: $\beta(x, t) = \exp_p(tx)$ where $p = \pi(x)$.

Lemma 2.14. *If $\beta : \mathcal{N} \times (-\varepsilon, \varepsilon) \rightarrow \mathcal{U}$ is a blow-up model for the neighborhood \mathcal{U} of B in (M, ω) , then the pull-back form $\beta^*\omega$ is an origami form whose null foliation is the circle fibration $\pi : \mathcal{N} \times \{0\} \rightarrow B$.*

Proof. By properties (1) and (2) of β , the form $\beta^*\omega$ is symplectic away from $\mathcal{N} \times \{0\}$. By property (3), on $\mathcal{N} \times \{0\}$ the kernel of $\beta^*\omega$ has dimension 2 and is fibrating. By property (4) the top power of $\beta^*\omega$ intersects zero transversally. \square

Let (M, ω) be a symplectic manifold with a codimension-two symplectic submanifold B .

Definition 2.15. A **model involution** of a tubular neighborhood \mathcal{U} of B is a symplectic involution $\gamma : \mathcal{U} \rightarrow \mathcal{U}$ preserving B such that on the connected components \mathcal{U}_i of \mathcal{U} where $\gamma(\mathcal{U}_i) = \mathcal{U}_i$ we have $\gamma|_{\mathcal{U}_i} = \text{id}_{\mathcal{U}_i}$.

A model involution γ induces a bundle map $\Gamma : \mathcal{N} \rightarrow \mathcal{N}$ covering $\gamma|_B$ by the formula

$$\Gamma[v] = [d\gamma_p(v)] \quad , \quad \text{for } v \in T_p M, p \in B \quad .$$

This is well-defined because $\gamma(B) = B$.

Remark 2.16. When B is the disjoint union of B_1 and B_2 , and correspondingly $\mathcal{U} = \mathcal{U}_1 \sqcup \mathcal{U}_2$, if $\gamma(B_1) = B_2$ then

$$\gamma_1 := \gamma|_{\mathcal{U}_1} : \mathcal{U}_1 \rightarrow \mathcal{U}_2 \quad \text{and} \quad \gamma|_{\mathcal{U}_2} = \gamma_1^{-1} : \mathcal{U}_2 \rightarrow \mathcal{U}_1 \quad .$$

In this case, $B/\gamma \simeq B_1$ and $\mathcal{N}/-\Gamma \simeq \mathcal{N}_1$ is the projectivised normal bundle to B_1 . \diamond

Proposition 2.17. *Let (M, ω) be a symplectic manifold, B a compact codimension-two symplectic submanifold and \mathcal{N} its projectivised normal bundle. Let $\gamma : \mathcal{U} \rightarrow \mathcal{U}$ be a model involution of a tubular neighborhood \mathcal{U} of B and $\Gamma : \mathcal{N} \rightarrow \mathcal{N}$ the induced bundle map.*

Then there is a natural origami manifold $(\widetilde{M}, \widetilde{\omega})$ with folding hypersurface diffeomorphic to $\mathcal{N}/-\Gamma$ and null fibration isomorphic to $\mathcal{N}/-\Gamma \rightarrow B/\gamma$.

Proof. Choose $\beta : \mathcal{N} \times (-\varepsilon, \varepsilon) \rightarrow \mathcal{U}$ a blow-up model for the neighborhood \mathcal{U} such that $\gamma \circ \beta = \beta \circ \Gamma$. This is always possible: for components \mathcal{U}_i of \mathcal{U} where $\gamma(\mathcal{U}_i) = \mathcal{U}_i$ this condition is trivial; for disjoint neighborhood components \mathcal{U}_i and \mathcal{U}_j such that $\gamma(\mathcal{U}_i) = \mathcal{U}_j$ (as in Remark 2.16), this condition amounts to choosing any blow-up model on one of these components and transporting it to the other by γ .

Then $\beta^*\omega$ is a folded symplectic form on $\mathcal{N} \times (-\varepsilon, \varepsilon)$ with folding hypersurface $\mathcal{N} \times \{0\}$ and null foliation integrating to the circle fibration $S^1 \hookrightarrow \mathcal{N} \xrightarrow{\pi} B$. We define

$$\widetilde{M} = \left(M \setminus B \bigcup \mathcal{N} \times (-\varepsilon, \varepsilon) \right) / \sim$$

where we quotient by

$$(x, t) \sim \beta(x, t) \text{ for } t > 0 \quad \text{and} \quad (x, t) \sim (-\Gamma(x), -t) \quad .$$

The forms ω on $M \setminus B$ and $\beta^*\omega$ on $\mathcal{N} \times (-\varepsilon, \varepsilon)$ induce on \widetilde{M} an origami form $\widetilde{\omega}$ with folding hypersurface $\mathcal{N}/-\Gamma$. Indeed β is a symplectomorphism for $t > 0$, and $(-\Gamma, -\text{id})$ on $\mathcal{N} \times (-\varepsilon, \varepsilon)$ is a symplectomorphism away from $t = 0$ (since β and γ are) and at points where $t = 0$ it is a local diffeomorphism. \square

Definition 2.18. The origami manifold $(\widetilde{M}, \widetilde{\omega})$ just constructed is called the **radial blow-up** of (M, ω) through (γ, B) .

When the starting manifold M is compact, the radial blow-up \widetilde{M} is also compact.

Example 2.19. Let M be a 2-sphere, B the union of two (distinct) points on it, and γ defined by a symplectomorphism from a Darboux neighborhood of one point to a Darboux neighborhood of the other. Then the radial blow-up \widetilde{M} is a Klein bottle and $\widetilde{\omega}$ a form which folds along a circle. \diamond

Example 2.20. Let M be a 2-sphere, B one point on it, and γ the identity map on a neighborhood of that point. Then the radial blow-up \widetilde{M} is \mathbb{RP}^2 and $\widetilde{\omega}$ a form which folds along a circle. \diamond

Remark 2.21. The quotient $\mathcal{N} \times (-\varepsilon, \varepsilon)/(-\Gamma, -\text{id})$ provides a collar neighborhood of the fold in $(\widetilde{M}, \widetilde{\omega})$.

When B splits into two disjoint components interchanged by γ as in Remark 2.16, this collar is orientable so the fold is coorientable. Example 2.19 illustrates a case where, even though the fold is coorientable, the radial blow-up $(\widetilde{M}, \widetilde{\omega})$ is not orientable.

When γ is the identity map, as in Example 2.20, the collar is nonorientable and the fold is not coorientable. In the latter case, the collar is a bundle of Möbius bands $S^1 \times (-\varepsilon, \varepsilon)/(-\text{id}, -\text{id})$ over B .

In general, γ will be the identity over some connected components of B and will interchange other components, so some components of the fold will be coorientable and others will not. \diamond

Remark 2.22. For the radial blow-up $(\widetilde{M}, \widetilde{\omega})$ to be orientable, the starting manifold (M, ω) must be the disjoint union of symplectic manifolds (M_1, ω_1) and (M_2, ω_2) with $B = B_1 \cup B_2$, $B_i \subset M_i$, such that $\gamma(B_1) = B_2$ as in Remark 2.16. In this case $(\widetilde{M}, \widetilde{\omega})$ may be equipped with an orientation such that

$$\widetilde{M}^+ \simeq M_1 \setminus B_1 \quad \text{and} \quad \widetilde{M}^- \simeq M_2 \setminus B_2 .$$

The folding hypersurface is diffeomorphic to \mathcal{N}_1 (or \mathcal{N}_2) and we have

$$\tilde{\omega} \simeq \begin{cases} \omega_1 & \text{on } M_1 \setminus B_1 \\ \omega_2 & \text{on } M_2 \setminus B_2 \\ \beta^* \omega_1 & \text{on } \mathcal{N} \times (-\varepsilon, \varepsilon) \end{cases}$$

We then say that $(\widetilde{M}, \tilde{\omega})$ is the blow-up of (M_1, ω_1) and (M_2, ω_2) through (γ_1, B_1) where γ_1 is the restriction of γ to a tubular neighborhood of B_1 . \diamond

Radial blow-up may be performed on an origami manifold at a symplectic submanifold B (away from the fold). When we start with two folded surfaces and radially blow them up at one point (away from the folding curves), topologically the resulting manifold is a connected sum at a point, $M_1 \# \overline{M_2}$, with all the previous folding curves plus a new closed curve.⁶

2.4. Cutting a radial blow-up.

Proposition 2.23. *Let (M, ω) be the radial blow-up of the symplectic manifolds (M_1, ω_1) and (M_2, ω_2) through (γ_1, B_1) where γ_1 is a symplectomorphism of tubular neighborhoods of codimension-two symplectic submanifolds B_1 and B_2 of M_1 and M_2 , respectively, taking B_1 to B_2 .*

Then the cutting of (M, ω) yields manifolds symplectomorphic to (M_1, ω_1) and (M_2, ω_2) where the symplectomorphisms carry B to B_1 and B_2 .

Proof. We first exhibit a symplectomorphism ρ_1 between the cut space (M_0^+, ω_0^+) of (M, ω) and the original manifold (M_1, ω_1) .

Let \mathcal{N} be the projectivised normal bundle to B_1 in M_1 and let $\beta : \mathcal{N} \times (-\varepsilon, \varepsilon) \rightarrow \mathcal{U}_1$ be a blow-up model. The cut space M_0^+ is obtained gluing the reduced space

$$\mu^{-1}(0)/S^1 = \left\{ (x, s, z) \in Z \times [0, \varepsilon^2) \times \mathbb{C} \mid s = \frac{|z|^2}{2} \right\} / S^1$$

with the manifold

$$M_1 \setminus B_1$$

⁶Since all \mathbb{RP}^{2n} are folded symplectic manifolds, the standard real blow-up of a folded symplectic manifold at a point away from its folding hypersurface still admits a folded symplectic form, obtained by viewing this operation as a connected sum.

via the diffeomorphisms

$$\begin{aligned} \mathcal{N} \times (0, \varepsilon) &\longrightarrow \mu^{-1}(0)/S^1 & \text{and} & & \mathcal{N} \times (0, \varepsilon) &\longrightarrow \mathcal{U}_1 \setminus B_1 \\ (x, t) &\longmapsto [x, t^2, t\sqrt{2}] & & & (x, t) &\longmapsto \beta(x, t) \end{aligned}$$

i.e., the gluing is by the identification $[x, t^2, t\sqrt{2}] \sim \beta(x, t)$ for $t > 0$ over $\mathcal{U}_1 \setminus B_1$. The symplectic form ω_0^+ on M_0^+ is equal to the reduced symplectic form on $\mu^{-1}(0)/S^1$ and equal to ω_1 on $M_1 \setminus B_1$ (the gluing diffeomorphism $[x, t^2, t\sqrt{2}] \mapsto \beta(x, t)$ is a symplectomorphism).

We want to define a map $\rho_1 : M_1 \rightarrow M_0^+$ which is the identity on $M_1 \setminus B_1$ and on \mathcal{U}_1 is the composed diffeomorphism

$$\begin{aligned} \delta_1 : \mathcal{U}_1 &\longrightarrow (\mathcal{N} \times \mathbb{C})/S^1 &\longrightarrow \mu^{-1}(0)/S^1 \\ &[x, z] &\longmapsto [x, |z|^2, z\sqrt{2}] \end{aligned}$$

where the first arrow is the inverse of the bundle isomorphism given by the blow-up model. In order to show that ρ_1 is well-defined we need to verify that $u_1 \in \mathcal{U}_1 \setminus B_1$ is equivalent to its image $\delta_1(u_1) \in \mu^{-1}(0)/S^1 \setminus B$. Indeed u_1 must correspond to $[x, z] \in (\mathcal{N} \times \mathbb{C})/S^1$ with $z \neq 0$. We write z as $z = te^{i\theta}$ with $t > 0$. Since $[x, z] = [e^{i\theta}x, t]$, we have $u_1 = \beta(e^{i\theta}x, t)$ and $\delta_1(u_1) = [e^{i\theta}x, |t|^2, t\sqrt{2}]$. These two are equivalent under $\beta(x, t) \sim [x, t^2, t\sqrt{2}]$, so ρ_1 is well-defined.

Furthermore, M_1 and M_0^+ are symplectic manifolds equipped with a diffeomorphism which is a symplectomorphism on the common dense subset $M_1 \setminus B_1$. We conclude that M_1 and M_0^+ must be globally symplectomorphic.

Now we tackle (M_2, ω_2) and (M_0^-, ω_0^-) . The cut space M_0^- is obtained gluing the same reduced space $\mu^{-1}(0)/S^1$ with the manifold $M_2 \setminus B_2$ via the diffeomorphisms

$$\begin{aligned} \mathcal{N} \times (-\varepsilon, 0) &\longrightarrow \overline{\mu^{-1}(0)/S^1} \\ (x, t) &\longmapsto [x, t^2, t\sqrt{2}] \end{aligned}$$

and

$$\begin{aligned} \mathcal{N} \times (-\varepsilon, 0) &\longrightarrow \overline{\mathcal{U}_1 \setminus B_1} &\xrightarrow{\gamma} & \overline{\mathcal{U}_2 \setminus B_2} \\ (x, t) &\longmapsto \beta(x, t) &\longmapsto & \gamma(\beta(x, t)) \end{aligned}$$

i.e., the gluing is by the identification $[x, t^2, t\sqrt{2}] \sim \gamma(\beta(x, t))$ for $t < 0$ over $\mathcal{U}_2 \setminus B_2$. The symplectic form ω_0^- on M_0^- restricts to the reduced form on $\mu^{-1}(0)/S^1$ and ω_2 on $M_2 \setminus B_2$.

We want to define a map $\rho_2 : M_2 \rightarrow M_0^-$ as being the identity on $M_2 \setminus B_2$ and on \mathcal{U}_2 being the composed diffeomorphism

$$\begin{aligned} \delta_2 : \mathcal{U}_2 &\xrightarrow{\gamma^{-1}} \mathcal{U}_1 \longrightarrow (\mathcal{N} \times \mathbb{C})/S^1 \longrightarrow \mu^{-1}(0)/S^1 \\ &[x, z] \longmapsto [x, |z|^2, z\sqrt{2}] \end{aligned}$$

where the second arrow is the inverse of the bundle isomorphism given by the blow-up model. In order to show that ρ_2 is well-defined we need to verify that $u_2 = \gamma(u_1) \in \mathcal{U}_2 \setminus B_2$ is equivalent to its image $\delta_2(u_2) \in \mu^{-1}(0)/S^1 \setminus B$. Indeed u_1 must correspond to $[x, z] \in (\mathcal{N} \times \mathbb{C})/S^1$ with $z \neq 0$. We write z as $z = -te^{i\theta}$ with $t < 0$. From $[x, z] = [-e^{i\theta}x, t]$, we conclude that $u_2 = \gamma(\beta(-e^{i\theta}x, t))$ and $\delta_2(u_2) = [-e^{i\theta}x, |t|^2, t\sqrt{2}]$. These two are equivalent under $\gamma(\beta(x, t)) \sim [x, t^2, t\sqrt{2}]$, so ρ_2 is well-defined.

As before, we conclude that M_2 and M_0^- must be globally symplectomorphic. \square

Lemma 2.24. *Let (M, ω) be the blow-up of the symplectic manifold (M_s, ω_s) through (γ, B) . We write $B = B_0 \sqcup B_1 \sqcup B_2$ and the domain of γ as $\mathcal{U} = \mathcal{U}_0 \sqcup \mathcal{U}_1 \sqcup \mathcal{U}_2$ where γ is the identity map on \mathcal{U}_0 and exchanges \mathcal{U}_1 and \mathcal{U}_2 .*

Let $(\overline{M}_s, \overline{\omega}_s)$ be the trivial double cover of (M_s, ω_s) with $\overline{B} = B^\uparrow \sqcup B^\downarrow$, $\overline{\mathcal{U}} = \mathcal{U}^\uparrow \sqcup \mathcal{U}^\downarrow$ the double covers of B and \mathcal{U} . Let $\overline{\gamma} : \overline{\mathcal{U}} \rightarrow \overline{\mathcal{U}}$ be the lift of γ satisfying

$$\overline{\gamma}(\mathcal{U}_0^\uparrow) = \mathcal{U}_0^\downarrow, \quad \overline{\gamma}(\mathcal{U}_1^\uparrow) = \mathcal{U}_2^\downarrow \quad \text{and} \quad \overline{\gamma}(\mathcal{U}_2^\uparrow) = \mathcal{U}_1^\downarrow$$

and let $(\overline{M}, \overline{\omega})$ be the blow-up of $(\overline{M}_s, \overline{\omega}_s)$ through $(\overline{\gamma}, \overline{B})$.

Then $(\overline{M}, \overline{\omega})$ is an orientable double cover of (M, ω) .

Proof. Being the double cover of an oriented manifold, we write

$$\overline{M}_s = M_s^\uparrow \sqcup M_s^\downarrow$$

with each component diffeomorphic to M_s . By Remark 2.22, the blow-up $(\overline{M}, \overline{\omega})$ is orientable and has

$$\overline{M}^+ \simeq M_s^\uparrow \setminus B^\uparrow \quad \text{and} \quad \overline{M}^- \simeq M_s^\downarrow \setminus B^\downarrow$$

with fold \mathcal{N}^\uparrow fibering over B^\uparrow . There is a natural two-to-one smooth projection $\overline{M} \rightarrow M$ taking $M_s^\uparrow \setminus B^\uparrow$ and $M_s^\downarrow \setminus B^\downarrow$ each diffeomorphically to $M \setminus Z$ where Z is the fold of (M, ω) , and taking the fold $\mathcal{N}^\uparrow \simeq \mathcal{N}$

of $(\overline{M}, \overline{\omega})$ to $Z \simeq \mathcal{N} / -\Gamma$ with $\Gamma : \mathcal{N} \rightarrow \mathcal{N}$ the bundle map induced by γ (the map $-\Gamma$ having no fixed points). \square

Corollary 2.25. *Let (M, ω) be the radial blow-up of the symplectic manifold (M_s, ω_s) through (γ, B) . Then the cutting of (M, ω) yields a manifold symplectomorphic to (M_s, ω_s) where the symplectomorphism carries the base to B .*

Proof. Let (M_{cut}, ω_{cut}) be the symplectic cut space of (M, ω) . Let $(\overline{M_s}, \overline{\omega_s})$ and $(\overline{M_{cut}}, \overline{\omega_{cut}})$ be the trivial double covers of (M_s, ω_s) and (M_{cut}, ω_{cut}) . By Lemma 2.24, the radial blow-up $(\overline{M}, \overline{\omega})$ of $(\overline{M_s}, \overline{\omega_s})$ through $(\overline{\gamma}, \overline{B})$ is an orientable double cover of (M, ω) . As a consequence of Definition 2.11, $(\overline{M_{cut}}, \overline{\omega_{cut}})$ is the symplectic cut space of $(\overline{M}, \overline{\omega})$. By Proposition 2.23, $(\overline{M_s}, \overline{\omega_s})$ and $(\overline{M_{cut}}, \overline{\omega_{cut}})$ are symplectomorphic relative to the centers. It follows that (M_s, ω_s) and (M_{cut}, ω_{cut}) are symplectomorphic relative to the centers. \square

2.5. Radially blowing-up cut pieces.

Proposition 2.26. *Let (M, ω) be an oriented origami manifold with null fibration $S^1 \hookrightarrow Z \xrightarrow{\pi} B$.*

Let (M_1, ω_1) and (M_2, ω_2) be its symplectic cut pieces, B_1 and B_2 the natural symplectic embedded images of B in each and $\gamma_1 : \mathcal{U}_1 \rightarrow \mathcal{U}_2$ the natural symplectomorphism of tubular neighborhoods of B_1 and B_2 as in Remark 2.9.

Let $(\widetilde{M}, \widetilde{\omega})$ be the radial blow-up of (M_1, ω_1) and (M_2, ω_2) through (γ_1, B_1) .

Then (M, ω) and $(\widetilde{M}, \widetilde{\omega})$ are equivalent origami manifolds.

Proof. Let $\varphi : Z \times (-\varepsilon, \varepsilon) \rightarrow \mathcal{U}$ be a Moser model for a tubular neighborhood \mathcal{U} of Z in M as in the proof of Proposition 2.8. Let \mathcal{N} be the projectivised normal bundle to B_1 in M_1 . By Proposition 2.8, the natural embedding of B in M_1 with image B_1 lifts to a bundle isomorphism from $\mathcal{N} \rightarrow B_1$ to $Z \rightarrow B$. Under this isomorphism, we pick the following natural blow-up model for the neighborhood $\mu^{-1}(0)/S^1$ of B_1 in (M_1, ω_1) :

$$\begin{aligned} \beta : Z \times (-\varepsilon, \varepsilon) &\longrightarrow \mu^{-1}(0)/S^1 \\ (x, t) &\longmapsto [x, t^2, t\sqrt{2}] . \end{aligned}$$

By recalling the construction of the reduced form ω_1 on $\mu^{-1}(0)/S^1$ (see proof of Proposition 2.8) we find that $\beta^*\omega_1 = \varphi^*\omega$. Hence in this case the origami manifold $(\widetilde{M}, \widetilde{\omega})$ has

$$\widetilde{M} = \left(M_1 \setminus B_1 \bigcup \overline{M_2 \setminus B_2} \bigcup Z \times (-\varepsilon, \varepsilon) \right) / \sim$$

where we quotient identifying

$$Z \times (0, \varepsilon) \xrightarrow{\beta} \mu^{-1}(0)/S^1 \subset \mathcal{U}_1 \setminus B_1$$

and

$$Z \times (-\varepsilon, 0) \xrightarrow{\beta} \overline{\mu^{-1}(0)/S^1} \xrightarrow{\gamma} \overline{\mu^{-1}(0)/S^1} \subset \overline{\mathcal{U}_2 \setminus B_2}$$

and we have

$$\widetilde{\omega} := \begin{cases} \omega_1 & \text{on } M_1 \setminus B_1 \\ \omega_2 & \text{on } M_2 \setminus B_2 \\ \beta^*\omega_1 & \text{on } Z \times (-\varepsilon, \varepsilon) . \end{cases}$$

The natural symplectomorphisms (from the proof of Proposition 2.8) $\overline{j^+} : M^+ \rightarrow M_1 \setminus B_1$ and $\overline{j^-} : M^- \rightarrow M_2 \setminus B_2$ extending $\varphi(x, t) \mapsto [x, t^2, t\sqrt{2}]$ make the following diagrams (one for $t > 0$, the other for $t < 0$) commute:

$$\begin{array}{ccc} M^+ \supset \mathcal{U}^+ & \xrightarrow{j^+} & \mu^{-1}(0)/S^1 \subset M_1 \setminus B_1 \\ \varphi \swarrow & & \nearrow \beta \\ & Z \times (0, \varepsilon) & \end{array}$$

and

$$\begin{array}{ccc} M^- \supset \mathcal{U}^- & \xrightarrow{j^-} & \overline{\mu^{-1}(0)/S^1} \subset \overline{M_2 \setminus B_2} \\ \varphi \swarrow & & \nearrow \beta \\ & Z \times (-\varepsilon, 0) & \end{array}$$

Therefore, the map $M \rightarrow \widetilde{M}$ defined by $\overline{j^+}$, $\overline{j^-}$ and φ^{-1} is a well-defined diffeomorphism pulling back $\widetilde{\omega}$ to ω . \square

Corollary 2.27. *Let (M, ω) be an origami manifold with null fibration $S^1 \hookrightarrow Z \xrightarrow{\pi} B$.*

Let $(M_{\text{cut}}, \omega_{\text{cut}})$ be its symplectic cut space, B_{cut} the natural symplectic embedded image of B in M_{cut} and $\gamma : \mathcal{U} \rightarrow \mathcal{U}$ a natural symplectomorphism of a tubular neighborhood \mathcal{U} of B_{cut} .

Let $(\widetilde{M}, \widetilde{\omega})$ be the radial blow-up of $(M_{\text{cut}}, \omega_{\text{cut}})$ through (γ, B_{cut}) .

Then (M, ω) and $(\widetilde{M}, \widetilde{\omega})$ are equivalent origami manifolds.

Proof. We pass to the orientable double covers. By Proposition 2.26, the orientable double cover of (M, ω) is equivalent to the blow-up of its cut space. By definition, the cut space of the double cover of (M, ω) is the double cover of (M_{cut}, ω_{cut}) . By Lemma 2.24, the blow-up of the latter double cover is the double cover of $(\widetilde{M}, \widetilde{\omega})$. \square

3. ORIGAMI POLYTOPES

3.1. Origami convexity.

Definition 3.1. If F_i is a face of a polytope Δ_i , $i = 1, 2$, in \mathbb{R}^n , we say that Δ_1 near F_1 **agrees** with Δ_2 near F_2 when $F_1 = F_2$ and there is an open subset \mathcal{U} of \mathbb{R}^n containing F_1 such that $\mathcal{U} \cap \Delta_1 = \mathcal{U} \cap \Delta_2$.

The following is an origami analogue of the Atiyah-Guillemin-Sternberg convexity theorem.

Theorem 3.2. *Let (M, ω, G, μ) be a connected compact origami manifold with null fibration $S^1 \hookrightarrow Z \xrightarrow{\pi} B$ and a hamiltonian action of an m -dimensional torus G with moment map $\mu : M \rightarrow \mathfrak{g}^*$. Then:*

- (a) *The image $\mu(M)$ of the moment map is the union of a finite number of convex polytopes Δ_i , $i = 1, \dots, N$, each of which is the image of the moment map restricted to the closure of a connected component of $M \setminus Z$.*
- (b) *Over each connected component Z' of Z , the null fibration is given by a subgroup of G if and only if $\mu(Z')$ is a facet of each of the one or two polytopes corresponding to the neighboring component(s) of $M \setminus Z$, and when those are two polytopes, they agree near the facet $\mu(Z')$.*

We call such images $\mu(M)$ **origami polytopes**.

Remark 3.3. When M is oriented, the facets from part (b) are always shared by *two* polytopes. In general, a component Z' is coorientable if and only if $\mu(Z')$ is a facet of two polytopes. \diamond

Proof.

- (a) Since the G -action preserves ω , it also preserves each connected component of the folding hypersurface Z and its null foliation

V . Choose an oriented trivializing section u of V . Average u so that it is G -invariant, i.e., replace it with

$$\frac{1}{|G|} \int_G g_* (u_{g^{-1}(p)}) dg .$$

Next, scale it uniformly over each orbit so that its integral curves all have period 2π , producing a vector field v which generates an action of S^1 on Z that commutes with the G -action. This S^1 -action also preserves the moment map μ : for any $X \in \mathfrak{g}$ with corresponding vector field $X^\#$ on M , we have over Z

$$\mathcal{L}_v \langle \mu, X \rangle = \iota_v d \langle \mu, X \rangle = -\iota_v \iota_{X^\#} \omega = \omega(v, X^\#) = 0 .$$

Using this v , the cutting construction from Section 2 has a hamiltonian version. Let (M_i, ω_i) , $i = 1, \dots, N$, be the resulting compact connected components of the symplectic cut space. Let B_i be the union of the components of $B = Z/S^1$ which naturally embed in M_i . Each $M_i \setminus B_i$ is symplectomorphic to a connected component $\mathcal{W}_i \subset M \setminus Z$ and M_i is the closure of $M_i \setminus B_i$. Each (M_i, ω_i) inherits a hamiltonian action of G with moment map μ_i which matches $\mu|_{\mathcal{W}_i}$ over $M_i \setminus B_i$ and is the well-defined S^1 -quotient of $\mu|_Z$ over B_i .

By the Atiyah-Guillemin-Sternberg convexity theorem [A, GS], each $\mu_i(M_i)$ is a convex polytope Δ_i . Since $\mu(M)$ is the union of the $\mu_i(M_i)$, we conclude that

$$\mu(M) = \bigcup_{i=1}^N \Delta_i .$$

(b) Assume first that M is orientable.

Let Z' be a connected component of Z with null fibration $Z' \rightarrow B'$. Let \mathcal{W}_1 and \mathcal{W}_2 be the two neighboring components of $M \setminus Z$ on each side of Z' , $(M_1, \omega_1, G, \mu_1)$ and $(M_2, \omega_2, G, \mu_2)$ the corresponding cut spaces with moment polytopes Δ_1 and Δ_2 .

Let \mathcal{U} be a G -invariant tubular neighborhood of Z' with a G -equivariant diffeomorphism $\varphi : Z' \times (-\varepsilon, \varepsilon) \rightarrow \mathcal{U}$ such that

$$\varphi^* \omega = p^* i^* \omega + d(t^2 p^* \alpha) ,$$

where G acts trivially on $(-\varepsilon, \varepsilon)$, $p : Z' \times (-\varepsilon, \varepsilon) \rightarrow Z'$ is the projection onto the first factor, t is the real coordinate on the interval $(-\varepsilon, \varepsilon)$ and α is a G -invariant S^1 -connection on Z' . The existence of such φ follows from an equivariant Moser trick, analogous to that in the proof of Proposition 2.8.

Without loss of generality, $Z' \times (0, \varepsilon)$ and $Z' \times (-\varepsilon, 0)$ correspond via φ to the two sides $\mathcal{U}_1 =: \mathcal{U} \cap \mathcal{W}_1$ and $\mathcal{U}_2 =: \mathcal{U} \cap \mathcal{W}_2$, respectively. The involution $\tau : \mathcal{U} \rightarrow \mathcal{U}$ translating $t \mapsto -t$ in $Z' \times (-\varepsilon, \varepsilon)$ is a G -equivariant (orientation-reversing) diffeomorphism preserving Z' , switching \mathcal{U}_1 and \mathcal{U}_2 but preserving ω . Hence the moment map satisfies $\mu \circ \tau = \mu$ and $\mu(\mathcal{U}_1) = \mu(\mathcal{U}_2)$.

When the null fibration is given by a subgroup of G , we cut the G -space \mathcal{U} at the level Z' . The image $\mu(Z')$ is the intersection of $\mu(\mathcal{U})$ with a hyperplane and thus a facet of both Δ_1 and Δ_2 .

Each $\mathcal{U}_i \cup B'$ is equivariantly symplectomorphic to a neighborhood \mathcal{V}_i of B' in $(M_i, \omega_i, G, \mu_i)$ with $\mu_i(\mathcal{V}_i) = \mu(\mathcal{U}_i) \cup \mu(Z')$, $i = 1, 2$. As a map to its image, the moment map is open [KB]. Since $\mu_1(\mathcal{V}_1) = \mu_2(\mathcal{V}_2)$, we conclude that Δ_1 and Δ_2 agree near the facet $\mu(Z')$.

For general null fibration, we cut the $G \times S^1$ -space \mathcal{U} with moment map (μ, t^2) at Z' , the S^1 -level $t^2 = 0$. The image of Z' by the $G \times S^1$ -moment map is the intersection of the image of the full \mathcal{U} with a hyperplane. We conclude that the image $\mu(Z')$ is the first factor projection $\pi : \mathfrak{g}^* \times \mathbb{R} \rightarrow \mathfrak{g}^*$ of a facet of a polytope $\tilde{\Delta}$ in $\mathfrak{g}^* \times \mathbb{R}$, so it can be of codimension zero or one; see Example 3.5.

If $\pi|_{\tilde{\Delta}} : \tilde{\Delta} \rightarrow \Delta_1$ is one-to-one, then facets of $\tilde{\Delta}$ map to facets of Δ_1 and $\tilde{\Delta}$ is contained in a hyperplane surjecting onto \mathfrak{g}^* . The normal to that hyperplane corresponds to a circle subgroup of $S^1 \times G$ acting trivially on \mathcal{U} and surjecting onto the S^1 -factor. This allows us to express the S^1 -action in terms of a subgroup of G .

If $\pi|_{\tilde{\Delta}} : \tilde{\Delta} \rightarrow \Delta_1$ is not one-to-one, it cannot map the facet $\tilde{F}_{Z'}$ of $\tilde{\Delta}$ corresponding to Z' to a facet of Δ_1 : otherwise, $\tilde{F}_{Z'}$ would contain nontrivial *vertical* vectors $(0, x) \in \mathfrak{g}^* \times \mathbb{R}$ which would forbid cutting. Hence, the normal to $\tilde{F}_{Z'}$ in $\tilde{\Delta}$ must be transverse to \mathfrak{g}^* , and the corresponding null fibration circle subgroup is not a subgroup of G .

When M is not necessarily orientable, we consider its orientable double cover and lift the hamiltonian torus action. The lifted moment map is the composition of the two-to-one projection with the original double map, and the result follows.

□

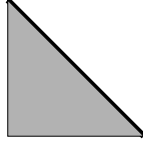
Example 3.4. Consider $(S^4, \omega_0, \mathbb{T}^2, \mu)$ where (S^4, ω_0) is a sphere as in Example 2.3 with \mathbb{T}^2 acting by

$$(e^{i\theta_1}, e^{i\theta_2}) \cdot \underbrace{(z_1, z_2, h)}_{\in \mathbb{C}^2 \times \mathbb{R} \simeq \mathbb{R}^5} = (e^{i\theta_1} z_1, e^{i\theta_2} z_2, h)$$

and moment map defined by

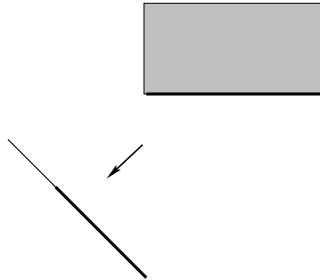
$$\mu(z_1, z_2, h) = \left(\underbrace{\frac{|z_1|^2}{2}}_{x_1}, \underbrace{\frac{|z_2|^2}{2}}_{x_2} \right)$$

whose image is the triangle $x_1 \geq 0$, $x_2 \geq 0$, $x_1 + x_2 \leq \frac{1}{2}$. The image $\mu(Z)$ of the folding hypersurface (the equator) is the hypotenuse.



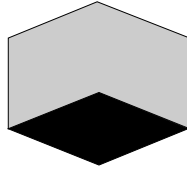
The null foliation is the Hopf fibration given by the diagonal circle subgroup of \mathbb{T}^2 . In this case, Theorem 3.2 says that the triangle is the union of two identical triangles, each of which is the moment polytope of one of the \mathbb{CP}^2 's obtained by cutting; see Example 2.6. Likewise, if $(S^4, \omega_0, \mathbb{T}^2, \mu)$ was blown-up at a pole, the triangle above would be the superposition of the same triangle with a trapezoid. \diamond

Example 3.5. Consider $(S^2 \times S^2, \omega_s \oplus \omega_f, S^1, \mu)$, where (S^2, ω_s) is a standard symplectic sphere, (S^2, ω_f) is a folded symplectic sphere with folding hypersurface given by a parallel, and S^1 acts as the diagonal of the standard rotation action of $S^1 \times S^1$ on the product manifold. Then the moment map image is a line segment and the image of the folding hypersurface is a nontrivial subsegment. Indeed, the image of μ is a 45° projection of the image of the moment map for the full $S^1 \times S^1$ action, i.e., a rectangle in which the folding hypersurface surjects to one of the sides.



By considering the first or second factors of $S^1 \times S^1$ alone, we get the two extreme cases in which the image of the folding hypersurface is either the full line segment or simply one of the boundary points.

The analogous six-dimensional examples $(S^2 \times S^2 \times S^2, \omega_s \oplus \omega_s \oplus \omega_f, \mathbb{T}^2, \mu)$ produce moment images which are rational projections of a cube, with the folding hypersurface mapped to rhombi.



3.2. Toric case.

Definition 3.6. A **toric origami manifold** (M, ω, G, μ) is a compact connected origami manifold (M, ω) equipped with an effective hamiltonian action of a torus G with $\dim G = \frac{1}{2} \dim M$ and with a choice of a corresponding moment map μ .

For a toric origami manifold (M, ω, G, μ) , orbits with trivial isotropy – the *principal orbits* – form a dense open subset of M [Br, p.179]. Any *coorientable* connected component Z' of Z has a G -invariant tubular neighborhood modelled on $Z' \times (-\varepsilon, \varepsilon)$ with a $G \times S^1$ hamiltonian action having moment map (μ, t^2) . As the orbits are isotropic submanifolds, the principal orbits of the $G \times S^1$ -action must still have dimension $\dim G$. Their stabilizer must be a one-dimensional compact connected subgroup surjecting onto S^1 . Hence, over those connected components of Z , the null fibration is given by a subgroup of G . A similar argument holds for *noncoorientable* connected components of Z , using orientable double covers. We have thus proven the following corollary of Theorem 3.2.

Corollary 3.7. *When (M, ω, G, μ) is a toric origami manifold, the moment map image of each connected component Z' of Z is a facet of each of the one or two polytopes corresponding to the neighboring component(s) of $M \setminus Z$, and when those are two polytopes, they agree near the facet $\mu(Z')$.*

Delzant spaces, also known as *symplectic toric manifolds*, are closed symplectic $2n$ -dimensional manifolds equipped with an effective hamiltonian action of an n -dimensional torus and with a corresponding moment map. Delzant's theorem [D] says that the image of the moment map (a polytope in \mathbb{R}^n) determines the Delzant space (up to an equivariant symplectomorphism intertwining the moment maps). The *Delzant conditions* on polytopes are conditions characterizing exactly those polytopes that occur as moment polytopes for Delzant spaces.⁷

Corollary 3.7 says that for a toric origami manifold (M, ω, G, μ) the image $\mu(M)$ is the superimposition of Delzant polytopes with certain compatibility conditions. Section 3.3 will show how all such (compatible) superimpositions occur and, in fact, classify toric origami manifolds.

For a Delzant space, G -equivariant symplectic neighborhoods of connected components of the orbit-type strata are simple to infer just by looking at the polytope.

Lemma 3.8. *Let $G = \mathbb{T}^n$ be an n -dimensional torus and $(M_i^{2n}, \omega_i, \mu_i)$, $i = 1, 2$, two symplectic toric manifolds. If the moment polytopes $\Delta_i := \mu_i(M_i)$ agree near facets $F_1 \subset \mu_1(M_1)$ and $F_2 \subset \mu_2(M_2)$, then there are G -invariant neighborhoods \mathcal{U}_i of $B_i = \mu_i^{-1}(F_i)$, $i = 1, 2$, with a G -equivariant symplectomorphism $\gamma : \mathcal{U}_1 \rightarrow \mathcal{U}_2$ extending a symplectomorphism $B_1 \rightarrow B_2$ and such that $\gamma^* \mu_2 = \mu_1$.*

Proof. Let \mathcal{U} be an open set containing $F_1 = F_2$ such that $\mathcal{U} \cap \Delta_1 = \mathcal{U} \cap \Delta_2$.

Perform symplectic cutting [L] on M_1 and M_2 by slicing Δ_i along a hyperplane parallel to F_i such that:

- the moment polytope $\tilde{\Delta}_i$ containing F_i is in the open set \mathcal{U} , and
- the hyperplane is close enough to F_i to guarantee that $\tilde{\Delta}_i$ satisfies the Delzant conditions.⁸

⁷A polytope in \mathbb{R}^n is **Delzant** if:

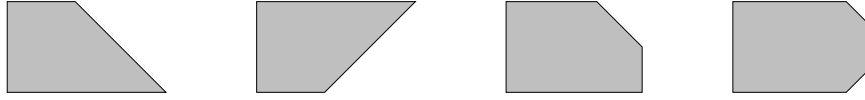
- there are n edges meeting at each vertex;
- each edge meeting at vertex p is of the form $p + tu_i$, $t \geq 0$, where $u_i \in \mathbb{Z}^n$;
- for each vertex, the corresponding u_1, \dots, u_n can be chosen to be a \mathbb{Z} -basis of \mathbb{Z}^n .

⁸For generic rational hyperplanes, the third of the Delzant conditions fails inasmuch as we only get a \mathbb{Q}^n -basis, thus we need to consider orbifolds.

Then $\widetilde{\Delta}_1 = \widetilde{\Delta}_2$. By Delzant's theorem, the corresponding cut spaces \widetilde{M}_1 and \widetilde{M}_2 are G -equivariantly symplectomorphic, the symplectomorphism pulling back one moment map to the other.

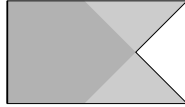
Since symplectic cutting is a local operation, restricting the previous symplectomorphism gives us a G -equivariant symplectomorphism between G -equivariant neighborhoods \mathcal{U}_i of B_i in M_i pulling back one moment map to the other. \square

Example 3.9. The following polytopes represent four different symplectic toric 4-manifolds: the topologically nontrivial S^2 -bundle over S^2 twice (Hirzebruch surfaces), an $S^2 \times S^2$ blown-up at one point and an $S^2 \times S^2$ blown-up at two points. If any two of these polytopes are translated so that their left vertical edges exactly superimpose, we get examples to which Lemma 3.8 applies (the relevant facets being the vertical facets on the left).



\diamond

Example 3.10. Let $(M_1, \omega_1, \mathbb{T}^2, \mu_1)$ and $(M_2, \omega_2, \mathbb{T}^2, \mu_2)$ be the first two symplectic toric manifolds from Example 3.9 (Hirzebruch surfaces). Let $(B, \omega_B, \mathbb{T}^2, \mu_B)$ be a symplectic S^2 with a hamiltonian (non-effective) \mathbb{T}^2 -action and hamiltonian embeddings j_i into $(M_i, \omega_i, \mathbb{T}^2, \mu_i)$ as preimages of the vertical facets. By Lemma 3.8, there exists a \mathbb{T}^2 -equivariant symplectomorphism $\gamma : \mathcal{U}_1 \rightarrow \mathcal{U}_2$ between invariant tubular neighborhoods \mathcal{U}_i of $j_i(B)$ extending a symplectomorphism $j_1(B) \rightarrow j_2(B)$ such that $\gamma^* \mu_2 = \mu_1$. The corresponding radial blow-up has the following origami polytope.



Different shades of grey distinguish regions where each point represents two orbits (darker) or one orbit (lighter), as results from the superimposition of two Hirzebruch polytopes.

This example may be considerably generalized; see Section 3.3. \diamond

Example 3.11. Dropping the origami hypothesis gives us exotic moment map images. For instance, take any symplectic toric manifold (M', ω', G, μ') , e.g. $S^2 \times S^2$, and use a regular closed curve inside the moment image to scoop out a G -invariant open subset corresponding to the region inside the curve. Let f be a defining function for the curve such that f is positive on the exterior. Consider the manifold

$$M = \{(p, x) \in M' \times \mathbb{R} \mid x^2 = f(p)\} .$$

This is naturally a *toric* folded symplectic manifold. However, the null foliation on Z is not fibrating: at points where the slope of the curve is irrational, the corresponding leaf is not compact.

For instance, when $M' = S^2 \times S^2$ and we take some closed curve, the moment map image is as on the left. If instead we discard the region corresponding to the outside of the curve (by choosing a function f positive on the interior of the curve), the moment map image is as on the right.



◇

3.3. Classification of toric origami manifolds. Let (M, ω, G, μ) be a toric origami manifold. By Theorem 3.2, the image $\mu(M)$ is the superimposition of the Delzant polytopes corresponding to the connected components of its symplectic cut space. Moreover, μ maps the folding hypersurface to certain facets possibly shared by two polytopes which agree near those facets.

Conversely, we will see that, given a *template* of an allowable superimposition of Delzant polytopes, we can construct a toric origami manifold whose moment image is that superimposition. Moreover, such templates classify toric origami manifolds.

Definition 3.12. An n -dimensional **origami template** is a pair $(\mathcal{P}, \mathcal{F})$, where \mathcal{P} is a (nonempty) finite collection of n -dimensional Delzant polytopes and \mathcal{F} is a collection of facets and pairs of facets of polytopes in \mathcal{P} satisfying the following properties:

- (a) for each pair $\{F_1, F_2\} \in \mathcal{F}$, the corresponding polytopes in \mathcal{P} agree near those facets;
- (b) if a facet F occurs in \mathcal{F} , either by itself or as a member of a pair, then neither F nor any of its neighbors occurs elsewhere in \mathcal{F} ;
- (c) the topological space constructed from the disjoint union $\sqcup \Delta_i$, $\Delta_i \in \mathcal{P}$, by identifying facet pairs in \mathcal{F} is connected.

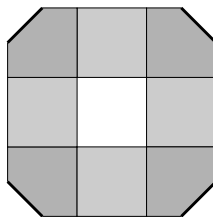
Theorem 3.13. *Toric origami manifolds are classified by origami templates up to equivariant equivalence preserving the moment maps. More specifically, there is a one-to-one correspondence*

$$\begin{aligned} \{2n\text{-diml toric origami manifolds}\} &\xrightarrow{1-1} \{n\text{-diml origami templates}\} \\ (M^{2n}, \omega, \mathbb{T}^n, \mu) &\longmapsto \mu(M). \end{aligned}$$

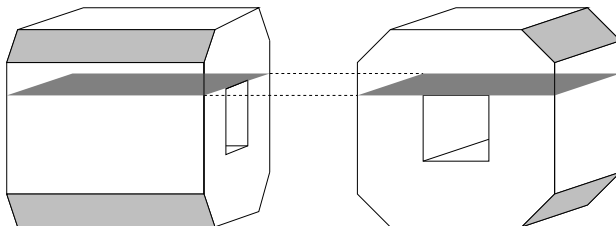
Proof. To build a toric origami manifold from a template $(\mathcal{P}, \mathcal{F})$, take the Delzant spaces corresponding to the Delzant polytopes in \mathcal{P} and radially blow up the inverse images of the facets occurring in sets in \mathcal{F} : for pairs $\{F_1, F_2\} \in \mathcal{F}$ the model involution γ uses the symplectomorphism from Lemma 3.8; for single faces $F \in \mathcal{F}$, the map γ must be the identity. The uniqueness part follows from an equivariant version of Corollary 2.27. \square

Remark 3.14. There is also a one-to-one correspondence between *oriented* origami toric manifolds and *oriented* origami templates. We say that an origami template is **oriented** when \mathcal{F} consists only of pairs and when the associated topological space is oriented as a manifold with corners; see Introduction. Indeed, when $\{F_1, F_2\} \in \mathcal{F}$, $F_1 \in \Delta_1$ and $F_2 \in \Delta_2$, the template orientation restricts to opposite orientations on the polytopes Δ_1 and Δ_2 inducing orientations on the corresponding components of $M \setminus Z$ which piece together to a global orientation of M , and vice-versa. \diamond

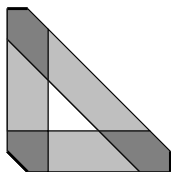
Example 3.15. Unlike ordinary toric manifolds, toric origami manifolds may come from non-simply connected templates. Let M be the manifold $S^2 \times S^2$ blown up at two points, with one S^2 factor having three times the area of the other: the associated polytope Δ is a rectangle with two corners removed. We can construct an origami template $(\mathcal{P}, \mathcal{F})$ where \mathcal{P} consists of four copies of Δ arranged in a square and \mathcal{F} is four pairs of edges coming from the blowups. The result is shown below. Note that the associated origami manifold is also not simply connected.



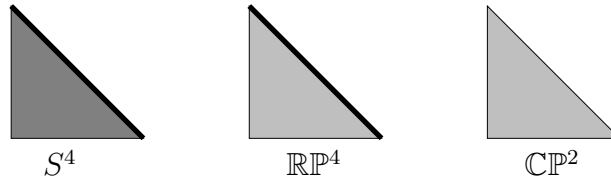
Example 3.16. We can form higher-dimensional analogues of the above example which fail to be k -connected for $k \geq 2$. In the case $k = 2$, for instance, let Δ' be the polytope associated to $M \times S^2$, and construct an origami template $(\mathcal{P}', \mathcal{F}')$ just as before: this gives the three-dimensional figures on the left and right below. We now superimpose these two solids along the dark shaded facets (the bottom facets of the top copies of Δ'), giving us a ninth pair of facets and the desired non-2-connected template.



Example 3.17. Although the facets of \mathcal{F} are necessarily paired if the origami manifold is oriented, the converse fails. As shown below, one can form a template of three polytopes, each corresponding to an $S^2 \times S^2$ blown up at two points, and three paired facets. Since each fold flips orientation, the resulting topological space is nonorientable.



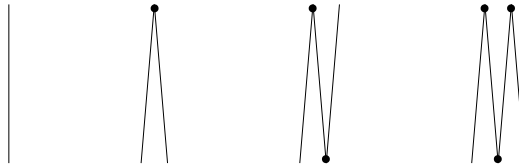
Example 3.18. Recall that the polytope associated to \mathbb{CP}^2 is a triangle (shown on the right below). The sphere S^4 (shown left) is the orientable toric origami manifold whose template is two copies of this triangle glued along one edge. Similarly, \mathbb{RP}^4 (shown center) is the nonorientable manifold whose template is a single copy of the triangle with a single folded edge. This exhibits S^4 as a double cover of \mathbb{RP}^4 at the level of templates.



◇

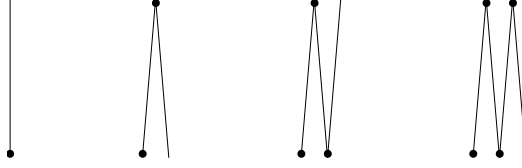
Example 3.19. We can classify all two-dimensional toric origami manifolds by classifying one-dimensional templates. These are disjoint unions of n segments connected at vertices with zero angle: all internal vertices are marked (for folds), and the endpoints (if they exist) may be marked. Each segment (resp. marking) gives a component of $M \setminus Z$ (resp. Z), while each unmarked endpoint corresponds to a fixed point. There are four families: ⁹

- Templates with two unmarked endpoints give manifolds diffeomorphic to S^2 : they have two fixed points and $n-1$ components of Z .

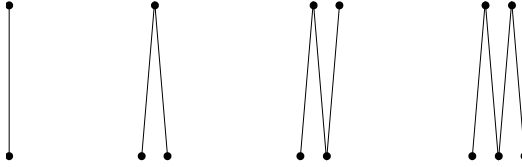


- Templates with one marked and one unmarked endpoint give manifolds diffeomorphic to \mathbb{RP}^2 : they have one fixed point and n components of Z .

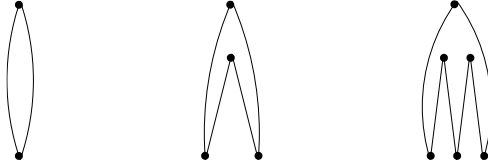
⁹Instead of drawing segments superimposed, we open up angles slightly to show the number of components. All pictures ignore segment lengths which account for continuous parameters of symplectic area in components of $M \setminus Z$.



- Templates with two marked endpoints give manifolds diffeomorphic to the Klein bottle: they have no fixed points and $n + 1$ components of Z .



- Templates with no endpoints give manifolds diffeomorphic to \mathbb{T}^2 : they have no fixed points and an even number n of components of Z .



◇

4. COBORDISM

We will now prove the following conjecture of Yael Karshon's.

Theorem 4.1. *An oriented origami manifold is cobordant to its symplectic cut space.*

Let (M, ω) be an oriented origami manifold with null fibration $S^1 \hookrightarrow Z \xrightarrow{\pi} B$ and assume for now that the folding hypersurface is connected. Extend the S^1 -action on Z to a tubular neighborhood \mathcal{U} of Z in a hamiltonian fashion (as in the proof of Proposition 2.8). Let $\mu : \mathcal{U} \rightarrow \mathbb{R}$ be a corresponding moment map with $\mu(Z) = 0$ and $\mu > 0$ elsewhere.

Let f be a non-negative function defined on M which agrees with μ on \mathcal{U} and is constant outside a larger neighborhood. Define the function

$$g : M \times \mathbb{C} \longrightarrow \mathbb{R}, \quad g(p, z) = |z|^2 - f(p).$$

Let $W_\delta = g^{-1}([-\delta, \delta])$. The S^1 -product action on $M \times \mathbb{C}$, where S^1 acts on \mathbb{C} by inverse rotation, induces an action of S^1 on W_δ which is free for δ sufficiently small.

Proposition 4.2. *The quotient space W_δ/S^1 is a manifold with boundary and its boundary is diffeomorphic to $M \sqcup M_0^+ \sqcup M_0^-$. In particular, M is cobordant to its symplectic cut space $M_0^+ \sqcup M_0^-$.*

Proof. The open set $g^{-1}((-2\delta, 2\delta))$ and the regular levels $g^{-1}(-\delta)$ and $g^{-1}(\delta)$ are submanifolds of $M \times \mathbb{C}$. Thus $W_\delta \subset g^{-1}((-2\delta, 2\delta))$ is a manifold with boundary $g^{-1}(-\delta) \sqcup g^{-1}(\delta)$ and, since S^1 acts freely on it, the quotient space W_δ/S^1 is a manifold with boundary $g^{-1}(-\delta)/S^1 \sqcup g^{-1}(\delta)/S^1$.

The natural diffeomorphism between M and $g^{-1}(\delta)/S^1$ is given by:

$$p \in M \longmapsto \left[p, \sqrt{f(p) + \delta} \right] \in g^{-1}(\delta)/S^1.$$

The remainder of the boundary, $g^{-1}(-\delta)/S^1$, has two disjoint components: $C^+ := g^{-1}(-\delta) \cap (M^+ \times \mathbb{C})/S^1$ and $C^- := g^{-1}(-\delta) \cap (M^- \times \mathbb{C})/S^1$. We will see that these are diffeomorphic to the cut pieces M_0^+ and M_0^- .

The proof mimics that of Proposition 2.8. Shrinking the tubular neighborhood \mathcal{U} as necessary, we have a Moser model $\varphi : Z \times (-\varepsilon, \varepsilon) \rightarrow \mathcal{U}$. The diffeomorphism $\psi : Z \times (0, \varepsilon^2) \rightarrow \mathcal{U}^+$ given by $\psi(x, s) = \varphi(x, \sqrt{s})$ induces a symplectic form $\nu := \psi^*\omega$ on $Z \times (0, \varepsilon^2)$ which can be extended naturally to $Z \times (-\varepsilon^2, \varepsilon^2)$. We form the product $(Z \times (-\varepsilon^2, \varepsilon^2), \nu) \times (\mathbb{C}, -\omega_0)$ endowed with an S^1 -action and corresponding moment map, and look at the δ -level, a codimension-one submanifold that decomposes as

$$\mu^{-1}(\delta) = Z \times \{\delta\} \times \{0\} \bigsqcup \{(x, s, z) \mid s > \delta, |z|^2 = 2(s - \delta)\}.$$

Since S^1 acts freely on each of the summands above, the quotient $\mu^{-1}(\delta)/S^1$ is a manifold, called the δ -cut space. The quotient $\mu^{-1}(\delta)/S^1$ may be viewed as the disjoint union $B \sqcup \mathcal{U}_\delta^+$, where $\mathcal{U}_\delta^+ = \varphi(Z \times (\delta, \varepsilon))$. Indeed, B embeds as a codimension-two submanifold via

$$\begin{aligned} j_\delta : \quad B &\longrightarrow \mu^{-1}(\delta)/S^1 \\ \pi(x) &\longmapsto [x, \delta, 0] \text{ for } x \in Z \end{aligned}$$

and \mathcal{U}_δ^+ embeds as an open dense submanifold via

$$\begin{aligned} j_\delta^+ : \quad \mathcal{U}_\delta^+ &\longrightarrow \mu^{-1}(\delta)/S^1 \\ \psi(x, s) &\longmapsto [x, s, \sqrt{2(s-\delta)}] . \end{aligned}$$

By gluing the rest of $M_{>\delta}^+ := \{p \in M^+ | f(p) > \delta\}$ to the δ -cut space along \mathcal{U}_δ^+ , we produce a manifold M_δ^+ . We will see that, shrinking δ if necessary, M_δ^+ is diffeomorphic to both M_0^+ and C^+ . Therefore, $M_0^+ \simeq C^+$; similarly, $M_0^- \simeq C^-$, yielding the desired result.

First, we define a diffeomorphism between M_0^+ and M_δ^+ by:

$$\begin{array}{llll} \frac{\mu^{-1}(0)}{S^1} \ni [x, s, \sqrt{2s}] & \mapsto & [x, s + h(s), \sqrt{2(s + h(s) - \delta)}] & \in \frac{\mu^{-1}(\delta)}{S^1} \\ \mathcal{U}^+ \ni \psi(x, s) & \mapsto & \psi(x, s + h(s)) & \in M_{>\delta}^+ \\ M^+ \setminus \mathcal{U}^+ \ni p & \mapsto & p & \in M_{>\delta}^+ \end{array}$$

where $h : \mathbb{R} \rightarrow \mathbb{R}$ is a non-increasing smooth function which satisfies $h(s) = \delta$ for $s \leq \frac{\varepsilon^2}{3}$ and $h(s) = 0$ for $s \geq \frac{2}{3}\varepsilon^2$. We can verify that the first branch is smooth by rewriting the map as

$$[x, s, z] \longmapsto [x, s + h(s), Q(s)z]$$

where $Q(0) = 1$ and $Q(s) = \sqrt{1 - \frac{\delta - h(s)}{s}}$ for $s \neq 0$. For δ sufficiently small (specifically $\delta < \frac{\varepsilon^2}{3}$), the function $Q(s)$ is everywhere defined and smooth. The embeddings j^+ and j_δ^+ intertwine the first and second branches.

Now we define a diffeomorphism between M_δ^+ and the boundary component C^+ by:

$$\begin{array}{llll} \mu^{-1}(\delta)/S^1 \ni [x, s, \sqrt{2(s-\delta)}] & \longmapsto & [\psi(x, s), \sqrt{s-\delta}] \\ M_{>\delta}^+ \ni p & \longmapsto & [p, \sqrt{f(p)-\delta}] \end{array}$$

Again, we verify that the first branch is smooth by rewriting the map as

$$[x, s, z] \longmapsto \left[\psi(x, s), \frac{z}{\sqrt{2}} \right] .$$

The two branches agree under j_δ^+ . □

The proof above does not depend on M_0^+ and M_0^- being symplectic, so an inductive argument on the number of connected components of the folding hypersurface completes the proof of Theorem 4.1. Also, note that, if M is a G -manifold and G preserves ω , then this cobordism can be made G -equivariant.

Remark 4.3. In the nonorientable case, we would obtain an orbifold cobordism. However, this is not interesting, since any manifold M bounds an orbifold, $(M \times [-1, 1]) / \mathbb{Z}_2$. \diamond

We conclude by discussing two consequences of Theorem 4.1. We recall that if (M, ω) is an oriented pre-symplectic manifold admitting a stable-complex structure and if it is prequantizable, i.e., the cohomology class of ω is integral, then this determines a line bundle $\mathbb{L} \rightarrow M$ together with a twisted spin- \mathbb{C} Dirac operator $\not{D}_{\mathbb{L}}$. The **quantization** of M is defined to be the virtual vector space

$$(3) \quad \mathcal{Q}(M) = (\ker \not{D}_{\mathbb{L}}) - (\operatorname{coker} \not{D}_{\mathbb{L}}).$$

Moreover, if M is a G -space and the action of G preserves \mathbb{L} and \not{D} , then (3) becomes a virtual representation of G . Note that changing the orientation of M changes the virtual vector space $\mathcal{Q}(M)$ to $-\mathcal{Q}(M)$.

Now suppose that the origami manifold M in Theorem 4.1 is integral. Then so are M_0^+ and M_0^- [CGW] and, since cobordant spaces have the same quantization [GGK],

$$(4) \quad \mathcal{Q}(M) \simeq \mathcal{Q}(M_0^+) - \mathcal{Q}(M_0^-)$$

is an isomorphism of virtual vector spaces, and, in the equivariant case, of virtual G -representations. The semi-classical analogue of this result for Duistermaat-Heckman measures (which will be defined in the next section) is again a corollary of Theorem 4.1: if \mathfrak{m} is the Duistermaat-Heckman measure of M and \mathfrak{m}_+ , \mathfrak{m}_- those of M_0^+ , M_0^- respectively, then

$$\mathfrak{m} = \mathfrak{m}_+ - \mathfrak{m}_-.$$

For oriented *toric* origami manifolds these results imply that \mathfrak{m} and $\mathcal{Q}(M)$ can be computed directly from the defining template of M . Let M_i , $i = 1, \dots, N$ be the connected components of $M_0^+ \sqcup M_0^-$. Using (4) we find that

$$\mathcal{Q}(M) \simeq \oplus (-1)^{\sigma_i} \mathcal{Q}(M_i)$$

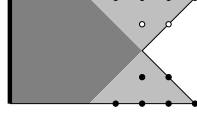
where $(-1)^{\sigma_i} = +1$ for $M_i \subset M_0^+$ and $(-1)^{\sigma_i} = -1$ for $M_i \subset M_0^-$. Moreover, if $\Delta_i \subset \mathcal{P}$ is the moment polytope of M_i then

$$\dim \mathcal{Q}(M_i) = \#(\mathbb{Z}_G^* \cap \Delta_i)$$

where $\mathbb{Z}_G^* \subset \mathfrak{g}^*$ is the weight lattice of G .

Example 4.4. Let M be the toric origami manifold in Example 3.10. The circles in the figure below mark the lattice points in its moment image with nonvanishing contribution to $\dim \mathcal{Q}(M)$: \bullet for $\sum (-1)^{\sigma_i} =$

+1 weights and \circ for $\sum(-1)^{\sigma_i} = -1$ weights. On lattice points where the two polytopes superimpose, the contributions from each polytope to the virtual dimension of the quantization of M cancel each other out, so no circle was drawn.



◇

For the Duistermaat-Heckman measure, the analogue of this result is even simpler:

$$\mathbf{m} = \sum (-1)^{\sigma_i} \mathbf{m}_{\Delta_i}$$

where \mathbf{m}_{Δ_i} is the standard Lebesgue measure.¹⁰

5. ORIGAMI VIA WEIGHT CONES

Let M^{2n} be an oriented compact G -manifold and $\tilde{\omega} \in \Omega_G^2(M)$ a closed equivariant 2-form. Then we can write $\tilde{\omega} = \omega + \mu$, where ω is a G -invariant 2-form and $\mu : M \rightarrow \mathfrak{g}^*$ is a corresponding moment map. The Duistermaat-Heckman measure associated with this data is a measure \mathbf{m} on \mathfrak{g}^* with the defining property

$$\int f \mathbf{m} = \int_M \mu^* f \omega^{2n}.$$

Notice that this is well-defined thanks to the orientation of M . If M^G is finite, this measure can be computed as follows: for each $p \in M$, let $\alpha_{i,p}$, $i = 1, \dots, n$ be the weights of isotropy representation of G on $T_p M$. Since M is not assumed to be complex (or even almost complex) these weights are only defined up to sign so we normalize them by “polarization”: we fix a $v \in \mathfrak{g}$ such that $\alpha_{i,p}(v) \neq 0$ for $i = 1, \dots, n$ and then require $\alpha_{i,p}(v) > 0$. This orientation of the $\alpha_{i,p}$ ’s induces an orientation of $T_p M$ and we define $(-1)^{\sigma_p} = \pm 1$ to be +1 if this orientation agrees with the given orientation of $T_p M$ and -1 otherwise.

¹⁰Pick a lattice basis e_1, \dots, e_n of \mathbb{Z}_G^* . The map $\mathbb{R}^n \rightarrow \mathfrak{g}^*$ given by $(t_1, \dots, t_n) \mapsto \sum t_i e_i$ maps Lebesgue measure on \mathbb{R}^n onto Lebesgue measure on \mathfrak{g}^* , and \mathbf{m}_{Δ_i} is the restriction of this measure to Δ_i .

Now let $L_p : \mathbb{R}^n \rightarrow \mathfrak{g}^*$ be the map

$$L_p(t) = \sum t_i \alpha_{i,p} + \mu(p).$$

Since the $\alpha_{i,p}$'s are polarized this map is proper and its image is a polyhedral cone. Let

$$\mathfrak{m}_p = (L_p)_* dt_1 \dots dt_n$$

be the “push-forward” by μ of the linear map L_p . Then by [GLS, Theorem 3.3.3] one has for the Duistermaat-Heckman measure the formula

$$\mathfrak{m} = \sum_p (-1)^{\sigma_p} \mathfrak{m}_p.$$

Note what happens in the oriented *toric* origami case with ω the origami form and μ the usual moment map. In this case, the cobordism in theorem 4.1 implies that

$$\mathfrak{m} = \sum (-1)^{\sigma_i} \mathfrak{m}_{\Delta_i}$$

where the Δ_i 's are the oriented polytopes belonging to the origami template with orientation $(-1)^{\sigma_i}$ and Lebesgue measure \mathfrak{m}_i . On the other hand, \mathfrak{m}_p is the measure on the cone $C_p = L_p(\mathbb{R}_+^n)$; notice that this measure is itself a constant multiple of Lebesgue measure since $\dim \mathfrak{g}^* = \dim \mathbb{R}_+^n$ and $L_p : \mathbb{R}_+^n \rightarrow C_p$ is bijective.

The appeal of this result is that it provides another way of visualizing an origami pattern, namely as a superimposition of these weight cones C_p oriented by $(-1)^{\sigma_p}$. We conjecture that the relation between these two origami pictures may yield interesting combinatorics.

6. COHOMOLOGY OF TORIC ORIGAMI

In this section we assume connectedness of the folding hypersurface Z . This assumption is essential for the argument below: the more general case of a nonconnected folding hypersurface remains open.

Let $(M, \omega, \mathbb{T}, \mu)$ be a $2n$ -dimensional oriented toric origami manifold with null fibration $S^1 \hookrightarrow Z \xrightarrow{\pi} B$ and connected folding hypersurface Z . Let $S^1 \subset \mathbb{T}$ be the circle group generating the null fibration and $f : M \rightarrow \mathbb{R}$ a corresponding moment map with $f = 0$ on Z and $f > 0$ on $M \setminus Z$. Note that, on a tubular neighborhood of Z given by a Moser diffeomorphism $\varphi : Z \times (-\varepsilon, \varepsilon) \rightarrow \mathcal{U}$, the origami form is $\varphi^* \omega = p^* i^* \omega + d(t^2 p^* \alpha)$ and hence the moment map is $\varphi^* f(x, t) = \frac{t^2}{2}$.

Near the folding hypersurface Z , the function f is essentially $\frac{t^2}{2}$, and away from it, f restricts to a moment map on an honest symplectic manifold $M \setminus Z$, and is thus Morse-Bott. Furthermore, Z is a nondegenerate critical manifold of codimension one.

Define $g : M \rightarrow \mathbb{R}$ as

$$g = \begin{cases} \sqrt{f} & \text{on } M^+ \\ 0 & \text{on } Z \\ -\sqrt{f} & \text{on } M^- \end{cases}$$

We claim that g is Morse-Bott and its critical manifolds are those of f , excluding Z . But this follows easily from the fact that, on \mathcal{U} , we have $\varphi^*g = \frac{t}{\sqrt{2}}$, whose derivative never vanishes, while on $M \setminus Z$, dg vanishes if and only if df vanishes:

$$dg = \begin{cases} df/(2\sqrt{f}) & \text{on } M^+ \\ -df/(2\sqrt{f}) & \text{on } M^- \end{cases}$$

Morse(-Bott) theory then gives us the cohomology groups $H_{\mathbb{T}}^k(M)$ in terms of the cohomology groups of the critical manifolds of g , $X \subset M \setminus Z$:

$$H_{\mathbb{T}}^k(M) = \begin{cases} 0 & \text{if } k \text{ odd} \\ \sum_X H_{\mathbb{T}}^{k-r_X}(X) & \text{if } k \text{ even} \end{cases}$$

where $r_X = \text{Ind}(X, g)$ is the index of the critical manifold X with respect to the function g , and is $\text{Ind}(X, f)$ if $X \subset M^+$, and $2d - \text{Ind}(X, f)$ if $X \subset M^-$.

REFERENCES

- [A] M. Atiyah, Convexity and commuting Hamiltonians, *Bull. London Math. Soc.* **14** (1982), 1-15.
- [Ba] I. Baykur, Kahler decomposition of 4-manifolds *Algebr. Geom. Topol.* **6** (2006), 1239-1265.
- [vB] J. von Bergmann, Pseudoholomorphic maps into folded symplectic four-manifolds *Geom. & Topol.* **11** (2007), 1-45.
- [Br] G. Bredon, *Introduction to Compact Transformation Groups*, Pure and Applied Mathematics **46**, Academic Press, 1972.
- [C] A. Cannas da Silva, Fold forms for four-folds, preprint (2002).
- [CGW] A. Cannas da Silva, V. Guillemin, C. Woodward, On the unfolding of folded symplectic structures, *Math. Res. Lett.* **7** (2000), 35-53.
- [D] T. Delzant, Hamiltoniens périodiques et images convexes de l'application moment, *Bull. Soc. Math. France* **116** (1988), 15-339.
- [G] R. Gompf, A new construction of symplectic manifolds, *Ann. of Math.* **142** (1995), 527-595.

- [GGK] V. Guillemin, V. Ginzburg, Y. Karshon, *Moment Maps, Cobordisms, and Hamiltonian Group Actions*, with an Appendix by M. Braverman, American Mathematical Society, Providence, 2002.
- [GLS] V. Guillemin, E. Lerman, S. Sternberg, *Symplectic Fibrations and Multiplicity Diagrams*, Cambridge University Press, Cambridge, 1996.
- [GS] V. Guillemin, S. Sternberg, Convexity properties of the moment mapping, *Invent. Math.* **67** (1982), 491-513.
- [H] D. Huybrechts, Birational symplectic manifolds and their deformations, *J. Differential Geom.* **45** (1997), 488-513.
- [KB] Y. Karshon, C. Bjorndahl, Revisiting Tietze-Nakajima – local and global convexity for maps, [arXiv:math/0701745](https://arxiv.org/abs/math/0701745), to appear in the *Canad. J. Math.*
- [Lee] C. Lee, Folded Symplectic Toric Four-Manifolds, Ph.D. thesis, Univ. Illinois at Urbana-Champaign, 2009.
- [L] E. Lerman, Symplectic cuts, *Math. Res. Lett.* **2** (1995), 247-258.
- [M] J. Martinet, Sur les singularités des formes différentielles, *Ann. Inst. Fourier (Grenoble)* **20** (1970), 95-178.

DEPARTMENT OF MATHEMATICS, PRINCETON UNIVERSITY, PRINCETON, NJ 08544-1000, USA AND DEPARTAMENTO DE MATEMÁTICA, INSTITUTO SUPERIOR TÉCNICO, 1049-001 LISBOA, PORTUGAL

E-mail address: `acannas@math.princeton.edu`

DEPARTMENT OF MATHEMATICS, MASSACHUSETTS INSTITUTE OF TECHNOLOGY, 77 MASSACHUSETTS AVENUE, CAMBRIDGE, MA 02139-4307, USA

E-mail address: `vwg@math.mit.edu`

DEPARTMENT OF MATHEMATICS, MASSACHUSETTS INSTITUTE OF TECHNOLOGY, 77 MASSACHUSETTS AVENUE, CAMBRIDGE, MA 02139-4307, USA

E-mail address: `arita@math.mit.edu`